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Phase structure of the generalized two-dimensional Yang-Mills theory on sphere

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Abstract. We find a general expression for the free energy of $G(\phi) = \phi^{2k}$ generalized two-dimensional (2D) Yang-Mills theory in the strong ($A > A_c$) region for large N. We also show that in this region, the density function of Young tableau of these models is a three-cut problem. In the specific ϕ^6 model, we show that the theory has a third order phase transition, like the ϕ^2 (YM_2) and ϕ^4 models. We note problems for cases where $k \geq 4$, and at the end, examine the phase structure of the $\phi^2 + g\phi^4$ model for the $g \leq A/4$ region.

1 Introduction

The pure 2D Yang-Mills theory (YM_2) is defined by the Lagrangian $\mathrm{tr}(F^2)$ on a compact Riemann surface. In an equivalent formulation of this theory, one can use $i\mathrm{tr}(BF) + \mathrm{tr}(B^2)$ as the Lagrangian, where B is an auxiliary pseudo-scalar field in the adjoint representation of the gauge group. Path integration over field B leaves an effective Lagrangian of the form $\mathrm{tr}(F^2)$.

Pure YM_2 theory, as applied to a compact Riemann surface, is characterized by its invariance under area-preserving diffeomorphism and its lack of propagating degrees of freedom. These properties are not unique to the $i\mathrm{tr}(BF) + \mathrm{tr}(B^2)$ theory, but rather are shared by a wide class of theories, called the generalized Yang-Mills theories (gYM_2). These theories are defined by replacing the $\mathrm{tr}(B^2)$ term by an arbitrary class function $\Lambda(B)$ ([10]). Aside from those discussed in [1], there are at least two reasons to study gYM_2. The first is that the Wilson loop vacuum expectation value of gYM_2 obeys the famous area law behaviour, ([11]), and this behaviour is a signal of confinement, one of the most important unsolved problems of QCD. Second, the existence of the third-order phase transition in some of the gYM_2 theories (one case is studied in [5] and other examples will be studied in this paper) is another indication for the equivalence of YM_2 and gYM_2 as a 2D counterpart of the theory of strong interaction.

The partition function of gYM_2 has been calculated in at least three ways: by regarding the generalized Yang-Mills action as a perturbation of topological theory at zero area ([10]); by following Migdal's suggestion about the local factor of plaquettes (it has been shown that this generalization satisfies the necessary requirements) ([1]); and by a continuum approach, using the standard path integral method ([11]). The gYM_2 theories can be further coupled to fermions, thus obtaining QCD_2 and generalized QCD_2 theories ([1]). These theories have generated much interest in recent years. Phase structure, string interpretation and algebraic aspects of these theories are reviewed in [2].

In this paper we explore the phase structure of the gYM_2 theories. An early study of the phase transition of YM_2 in the large-N limit on a lattice reveals a third-order phase transition ([3]). The study of pure continuum YM_2 for large N on a sphere yields a similar result ([4]). This result is obtained by calculating free energy as a function of the area of the sphere (A) and distinguishing between the small- and large-area behaviour of this function. In [5], the authors consider gYM_2 for large N on a sphere and find an exact expression for an arbitrary gYM_2 theory in the weak ($A < A_c$) region, where A_c is the critical area. They also find a third-order phase transition for the specific ϕ^4 model.

In addition, we discuss the issue of phase transition for a wider class of theories. In Sect. 2 we review the derivation of the free energy and density function in the weak ($A < A_c$) region. In Sect. 3 we study the ϕ^{2k} theories. First, we show that the density function for these models has two maxima in the ($A < A_c$) region, like the ϕ^4 model. This enables us to use the method in [5] to obtain a general expression for free energy for these theories in the strong ($A > A_c$) region. In Sect. 4, we compute the free energy near the transition point for the specific ϕ^6 model, show that this model also has a third-order phase transition, and remark briefly on models where $k \geq 4$. Finally, in Sect. 5, we study another class of models, namely

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the $\phi^2+g\phi^4$ models. If $g \le A/4$, then the density function will have only one maximum at the origin. We show that these models also undergo a third-order phase transition in this domain.

2 Large-N behaviour of gYM$_2$ at $A < A_c$

The partition function of the gYM$_2$ on a sphere is [5]

$$Z = \sum_r d_r^2 e^{-A\Lambda(r)}, \qquad (1)$$

where r is the irreducible representation of the gauge group, d_r is the dimension of the r-th representation, A is the area of the sphere, and $\Lambda(r)$ is

$$\Lambda(r) = \sum_{k=1}^{p} \frac{a_k}{N^{k-1}} C_k(r), \qquad (2)$$

in which C_k is the k-th Casimir of the group, and a_k is an arbitrary constant. We parametrize the representation of the gauge group $U(N)$ by $n_1 \ge n_2 \ge \cdots \ge n_N$, where n_i is the length of the i-th row of the Young tableau. It is found that

$$d_r = \prod_{1\le i<j\le N} \left(1 + \frac{n_i - n_j}{j-i}\right),$$

and

$$C_k = \sum_{i=1}^{N} [(n_i + N - i)^k - (N-i)^k]. \qquad (3)$$

To make the partition function (1) convergent, it is necessary that p in (2) be even, and that $a_p > 0$.

Now, following [4], one can write the partition function (1), for large N, as a path integral over continuous parameters. We introduce the continuous function

$$\phi(x) = -n(x) - 1 + x, \qquad (4)$$

where

$$0 \le x := i/N \le 1 \quad \text{and} \quad n(x) := n_i/N. \qquad (5)$$

The partition function (1) then becomes

$$Z = \int \prod_{0\le x\le 1} \mathrm{d}\phi(x) e^{S[\phi(x)]}, \qquad (6)$$

where

$$S(\phi) = N^2\{-A\int_0^1 \mathrm{d}x G[\phi(x)] + \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \ \ \log|\phi(x)-\phi(y)| \ \ \}, \qquad (7)$$

apart from an unimportant constant, and

$$G[\phi] = \sum_{k=1}^{p} (-1)^k a_k \phi^k. \qquad (8)$$

Now we introduce the density

$$\rho[\phi(x)] = \frac{\mathrm{d}x}{\mathrm{d}\phi(x)}, \qquad (9)$$

where, for cases in which G is an even function, the normalization condition for ρ is

$$\int_{-a}^{a} \rho(\lambda)\mathrm{d}\lambda = 1. \qquad (10)$$

The function $\rho(z)$ in this case is ([5]),

$$\rho(z) = \frac{\sqrt{a^2-z^2}}{\pi} \times \sum_{n,q=0}^{\infty} \frac{(2n-1)!!}{2^n n!(2n+q+1)!} a^{2n} z^q g^{(2n+q+1)}(0), \qquad (11)$$

where

$$g(\phi) = \frac{A}{2} G'(\phi), \qquad (12)$$

and $g^{(n)}$ is the n-th derivative of g. Similarly, one can express the normalization condition, (10), as

$$\sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!(2n-1)!} a^{2n} g^{(2n-1)}(0) = 1. \qquad (13)$$

Defining the free energy as

$$F := -\frac{1}{N^2} \ln Z, \qquad (14)$$

the derivative of this free energy with respect to the area of the sphere is then

$$F'(A) = \int_0^1 \mathrm{d}x \ \ G[\phi(x)] = \int_{-a}^{a} \mathrm{d}\lambda \ \ G(\lambda)\rho(\lambda). \qquad (15)$$

The condition $n_1 \ge n_2 \ge \cdots \ge n_N$ imposes the following condition on the density $\rho(\lambda)$:

$$\rho(\lambda) \le 1. \qquad (16)$$

Thus, we first determine the parameter a from (13), then we calculate $F'(A)$ from (15). Note that the above solution is valid in the weak ($A < A_c$) region, where A_c is the critical area. If $A > A_c$, then the condition $\rho \le 1$ is violated.

3 The $G(\phi)=\phi^{2k}$ model

In order to study the behaviour of any model in the strong $(A > A_c)$ region, we need to know the explicit form of the density ρ in the weak $(A < A_c)$ region. From (11) we can obtain ρ for any even function $G(\phi)$. However, in this section we consider a simple case, namely the $G(\phi) = \phi^{2k}$ model with arbitrary positive integer k. The density function ρ in the weak region is

$$\rho(z) = \frac{kA}{\pi}\sqrt{a^2 - z^2}\sum_{n=0}^{k-1}\frac{(2n-1)!!}{2^n n!}a^{2n}z^{2k-2n-2}. \quad (17)$$

The interesting point is that the above density function has only one minimum at $z = 0$, and two maxima which are symmetric with respect to the origin. To see this, notice that setting the derivative of ρ equal to zero will yield $z = 0$, and

$$f(y) = -1 + \sum_{n=0}^{k-2}\frac{(2n-1)!!}{2^{n+1}(n+1)!}y^{-(n+1)} = 0, \quad (18)$$

where $y = z^2/a^2$. Because all of the coefficients of y in (18) are positive, the function $f(y)$ is a monotonically decreasing function and has only one root, i.e., y_0. Next, expanding ρ near the origin, we obtain

$$\rho(z) = \frac{kA}{\pi}\frac{(2k-3)!!}{2^{k-1}(k-1)!} \times a^{2k}\left(1 + \frac{2k-1}{2(2k-3)}z^2a^{-2} + \cdots\right), \quad (19)$$

thus, $\rho''(0) > 0$. The origin is then a minimum. But near the points $z = \pm a$, the curve $\rho(z)$ is concave downward; consequently the points $z_0^{(\pm)} = \pm a\sqrt{y_0}$ will correspond to two symmetric maxima of the density. Therefore in the strong region, all the ϕ^{2k} models are *three-cut* problems. The function $F'(A)$ for $G(\phi) = \phi^{2k}$ in the weak region is ([5]):

$$F'_w(A) = \frac{1}{2kA}. \quad (20)$$

Next, we study the strong $(A > A_c)$ region. Following [5], we use the following ansatz for ρ:

$$\rho_s(z) = \begin{cases} 1, & z \in [-b,-c]\bigcup[c,b] =: L' \\ \tilde{\rho}_s(z), & z \in [-a,-b]\bigcup[-c,c]\bigcup[b,a] =: L. \end{cases} \quad (21)$$

Then, if we define the function $H_s(z)$ as in [6],

$$H_s(z) := \mathrm{P}\int_{-a}^{a}\mathrm{d}w\ \frac{\rho_s(w)}{z-w}, \quad (22)$$

where P indicates the principal value of the integral, it has the following expansion for large values of z:

$$H_s(z) = \frac{1}{z} + \frac{1}{z^3}\int_{-a}^{a}\rho_s(\lambda)\lambda^2\mathrm{d}\lambda + \cdots + \frac{1}{z^{2k+1}}F'_s(A) + \cdots. \quad (23)$$

Hence, one can obtain $F'_s(A)$ via expansion of $H_s(z)$.

To calculate the function $H_s(z)$, we follow the same steps outlined in [5], and using some complex analysis techniques ([7]), obtain the following result for ϕ^{2k} model:

$$H_s(z) = kAz^{2k-1} - kAR(z)\sum_{n,p,q=0}^{\infty}\alpha(n,p,q)z^{2(k-n-p-q-2)} - 2R(z)\int_c^b\frac{\lambda\mathrm{d}\lambda}{(z^2-\lambda^2)R(\lambda)}. \quad (24)$$

where

$$\alpha(n,p,q) = \frac{(2n-1)!!(2p-1)!!(2q-1)!!}{2^{n+p+q}n!p!q!}a^{2n}b^{2p}c^{2q}, \quad (25)$$

and

$$R(z) = \sqrt{(z^2-a^2)(z^2-b^2)(z^2-c^2)}. \quad (26)$$

Now, we expand $H_s(z)/R(z)$ and demand that it behaves like $1/z^4$ for large values of z. It can be shown that the coefficients for all positive powers of z in the above expansion are equal to zero. Next, we calculate the coefficients of $1/z^2$; this gives us

$$kA\sum_{n,p,q=0}^{\infty}\alpha(n,p,q) - 2\int_c^b\frac{\lambda\mathrm{d}\lambda}{R(\lambda)} = 0, \quad (27)$$

in which $n+p+q = k-1$. By setting the coefficient of $1/z^4$ to one, we obtain

$$kA\sum_{n,p,q=0}^{\infty}\alpha(n,p,q) - 2\int_c^b\frac{\lambda^3\mathrm{d}\lambda}{R(\lambda)} = 1, \quad (28)$$

where $n+p+q = k$. In the $k = 2$ case, the ϕ^4 theory, (27) and (28) reduce to

$$A(a^2+b^2+c^2) = 2\int_c^b\frac{\lambda\mathrm{d}\lambda}{R(\lambda)}, \quad (29)$$

and

$$A\left[\ \frac{3}{4}(a^4+b^4+c^4) + \frac{1}{2}(a^2b^2+b^2c^2+c^2a^2)\ \right] - 2\int_c^b\frac{\lambda^3\mathrm{d}\lambda}{R(\lambda)} = 1, \quad (30)$$

which are the same equations that were obtained in [5].

We can express the action in terms of $\rho_s(z)$. If we maximize this action, along with the normalization condition, (10), as a constraint, we obtain another equation. This procedure is fully explained in [5,8]. The result is

$$kA\sum_{n,p,q=0}\int_c^b\alpha(n,p,q)z^{2(k-n-p-q-2)}R(z)\mathrm{d}z + 2\int_c^b\mathrm{d}z\ \ \mathrm{P}\int_c^b\ \frac{R(z)\lambda\mathrm{d}\lambda}{(z^2-\lambda^2)R(\lambda)} = 0. \quad (31)$$

Note that there are three unknown parameters, a, b, and c, in equations (27), (28), and (31).

To compute the function $F'_s(A)$, we start from the function $H_s(z)$ directly. First we expand $R(z)$:

$$R(z) = z^3\sqrt{(1-\frac{a^2}{z^2})(1-\frac{b^2}{z^2})(1-\frac{c^2}{z^2})}$$
$$= -z^3 \sum_{n',p',q'=0}^{\infty} \beta(n',p',q')z^{-2(n'+p'+q')}, \quad (32)$$

where

$$\beta(n,p,q) = \frac{(2n-3)!!(2p-3)!!(2q-3)!!}{2^{n+p+q}n!p!q!}a^{2n}b^{2p}c^{2q}. \quad (33)$$

Furthermore, we define $(-3)!! = -1$. If we substitute the above expansion into (24), we obtain

$$H_s(z) = kAz^{2k-1} + kA \times \sum_{n,p,q,n',p',q'=0} \frac{\alpha(n,p,q)\beta(n',p',q')}{z^{2(n+p+q-k)+2(n'+p'+q')+1}} + 2\sum_{n=0}^{\infty}\sum_{n',p',q'=0} \frac{\beta(n',p',q')}{z^{2(n+n'+p'+q')-1}} \int_c^b \frac{\lambda^{2n+1}\mathrm{d}\lambda}{R(\lambda)}. \quad (34)$$

From (23) we see that the coefficient of $1/z^{2k+1}$ in the expansion of $H_s(z)$ is $F'_s(A)$. Therefore, from (34) we get

$$F'_s(A) = kA \sum_{n,p,q,n',p',q'=0} \alpha(n,p,q)\beta(n',p',q') + 2\sum_{n,n',p',q'=0} \beta(n',p',q') \int_c^b \frac{\lambda^{2n+1}\mathrm{d}\lambda}{R(\lambda)}. \quad (35)$$

Additionally, in the first summation of (35) we have the following conditions on the indices:

$$(n+p+q) + (n'+p'+q') = 2k, \quad (36a)$$

and

$$2k - 2n - 2p - 2q - 4 \geq 0. \quad (36b)$$

Condition (36a) appears due to the selection of a specific power of z in the expansion, and condition (36b) is due to complex integration. Furthermore, in the second summation we have the follwing condition on the indices:

$$n + (n'+p'+q') = k+1. \quad (36c)$$

In this way, we find the explicit relation of the free energy of the ϕ^{2k} models. For the $k = 2$ case, our results agree with those in [5].

4 The $G(\phi) = \phi^6$ model

Applying the previous results, we will investigate carefully the phase structure of the $G(\phi) = \phi^6$ model. From (17) we obtain the density ρ for this model in the weak region; the result is

$$\rho(z) = \frac{3A}{\pi}\left(\frac{3a^4}{8} + \frac{a^2z^2}{2} + z^4\right)\sqrt{a^2 - z^2}. \quad (37)$$

From the normalization condition, (13), we obtain $a = (16/(15A))^{1/6}$. In addition, we see from (20) that $F'_w(A) = 1/(6A)$. This density function has a minimum at z=0, and two maxima at $z_0^{(\pm)} = \pm\left(\sqrt{\sqrt{3}+1}\right)a/2$. At $A = A_c$, the density function (37) is equal to one at $z_0^{(\pm)}$. From this, we find the critical area A_c:

$$A_c = \pi^6\left(\frac{3125}{10368} - \frac{15625\sqrt{3}}{93312}\right). \quad (38)$$

For $A > A_c$, (37) is not valid. Next, we analyse this model in the strong ($A > A_c$) region. Equations (27), (28), and (31) in this case become (39), (40), and (41), respectively:

$$3A\left[\frac{3}{8}(a^4+b^4+c^4) + \frac{1}{4}(a^2b^2+b^2c^2+c^2a^2)\right] - 2\int_c^b \frac{\lambda \mathrm{d}\lambda}{R(\lambda)} = 0, \quad (39)$$

$$3A\Big[\frac{5}{16}(a^6+b^6+c^6) + \frac{3}{16}(a^2b^4+a^2c^4+b^2a^4+b^2c^4+c^2a^4+c^2b^4) + \frac{1}{8}a^2b^2c^2\Big] - 2\int_c^b \frac{\lambda^3\mathrm{d}\lambda}{R(\lambda)} = 1, \quad (40)$$

and

$$3A\int_c^b \left(z^2 + \frac{a^2+b^2+c^2}{2}\right)R(z)\mathrm{d}z + 2\int_c^b \mathrm{d}z \ \mathrm{P}\int_c^b \frac{R(z)\lambda\mathrm{d}\lambda}{(z^2-\lambda^2)R(\lambda)} = 0. \quad (41)$$

To study the structure of the phase transition, we must consider the theory applied to a sphere with $A = A_c + \epsilon$ area, where ϵ is an infinitesimal positive number. In this region, following [5], we use the following change of variables:

$$c = s(1-y),$$
$$b = s(1+y), \quad (42)$$
$$a = s\sqrt{2\sqrt{3} - 2 + e}.$$

Note that these parameters are introduced so that at critical points, e and y are equal to zero and s is equal to z_0^+. Now, expanding the equations (39), (40) and (41), we find

$$
\begin{aligned}
&As^4(18-6\sqrt{3})-\frac{\pi}{\eta s}\\
&+\left(As^4\left(\frac{9\sqrt{3}}{2}-3\right)+\frac{\pi}{\eta s}\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)\right)e\\
&+\left(\frac{9As^4}{8}-\frac{\pi}{\eta s}\left(\frac{7}{8}+\frac{\sqrt{3}}{2}\right)\right)e^2\\
&+\left((9+3\sqrt{3})As^4-\frac{\pi}{\eta s}\left(\frac{9}{4}+\frac{4\sqrt{3}}{3}\right)\right)y^2\\
&+\left(\frac{3As^4}{2}+\frac{\pi}{\eta s}\left(\frac{77\sqrt{3}}{12}+\frac{89}{8}\right)\right)ey^2\\
&+\left(3As^4-\frac{\pi}{\eta s}\left(\frac{5363}{192}+\frac{129\sqrt{3}}{8}\right)\right)y^4=0, \quad (43)
\end{aligned}
$$

and

$$
\begin{aligned}
&(39\sqrt{3}-57)As^6-1-\frac{\pi s}{\eta} \qquad (44)\\
&+\left((42-18\sqrt{3})As^6+\frac{\pi s}{\eta}\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)\right)e\\
&+\left(\left(\frac{45\sqrt{3}}{8}-\frac{9}{2}\right)As^6-\frac{\pi s}{\eta}\left(\frac{7}{8}+\frac{\sqrt{3}}{2}\right)\right)e^2\\
&+\left((33+3\sqrt{3})As^6-\frac{\pi s}{\eta}\left(2\sqrt{3}+\frac{17}{4}\right)\right)y^2\\
&+\left(\left(\frac{3}{2}+\frac{9\sqrt{3}}{2}\right)As^6+\frac{\pi s}{\eta}\left(\frac{35\sqrt{3}}{4}+\frac{121}{8}\right)\right)ey^2\\
&+\left((3\sqrt{3}+24)As^6-\frac{\pi s}{\eta}\left(\frac{199\sqrt{3}}{8}+\frac{8267}{192}\right)\right)y^4=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&3(1+\sqrt{3})As^5-\frac{\pi}{\eta}\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)\\
&+\left(\left(\frac{5\sqrt{3}}{2}+6\right)As^5+\frac{\pi}{\eta}\left(\frac{7}{6}+\frac{2\sqrt{3}}{3}\right)\right)e\\
&-\left(\left(\frac{7\sqrt{3}}{8}+\frac{13}{8}\right)As^5+\frac{\pi}{\eta}\left(\frac{5}{2}+\frac{13\sqrt{3}}{9}\right)\right)e^2\\
&-\left((2+4\sqrt{3})As^5+\frac{\pi}{\eta}\left(\frac{91\sqrt{3}}{36}+\frac{211}{48}\right)\right)y^2\\
&+\left(\left(\frac{25}{2}+\frac{15\sqrt{3}}{2}\right)As^5+\frac{\pi}{\eta}\left(\frac{635}{24}+\frac{275\sqrt{3}}{18}\right)\right)ey^2\\
&-\left(\left(\frac{56}{3}+\frac{32\sqrt{3}}{3}\right)\right.\\
&\left.+\frac{\pi}{\eta}\left(\frac{60673\sqrt{3}}{1728}+\frac{23353}{384}\right)\right)y^4=0. \qquad (45)
\end{aligned}
$$

The parameter $\eta=\sqrt{2\sqrt{3}-3}$ is used for the sake of brevity in the above formulas. We also have kept terms up to order y^4 or e^2 (we will show that e is of order y^2). Next we obtain s from (43); the result is

$$
\begin{aligned}
s=&\left(\frac{\pi}{A}\right)^{\frac{1}{5}}\left(\frac{12+7\sqrt{3}}{648}\right)^{\frac{1}{10}}\left[1-\left(\frac{1+\sqrt{3}}{8}\right)e\right. \qquad (46)\\
&+\left(\frac{5}{32}+\frac{\sqrt{3}}{12}\right)e^2+\left(\frac{1}{4}+\frac{\sqrt{3}}{6}\right)y^2\\
&\left.-\left(\frac{167}{96}+\frac{95\sqrt{3}}{96}\right)ey^2+\left(\frac{911}{192}+\frac{11\sqrt{3}}{4}\right)y^4\right].
\end{aligned}
$$

Substituting s in (45) gives us

$$
e=\left(\frac{5\sqrt{3}}{2}-\frac{1}{2}\right)y^2-\left(\frac{15}{16}+\frac{37\sqrt{3}}{48}\right)y^4. \qquad (47)
$$

So e is of order y^2. Using (44), we obtain

$$
y^2=\left(\frac{4\sqrt{3}}{15}-\frac{2}{5}\right)\delta+\left(\frac{317}{360}-\frac{77\sqrt{3}}{150}\right)\delta^2, \qquad (48)
$$

and

$$
e=\left(\frac{11}{5}-\frac{17\sqrt{3}}{15}\right)\delta+\left(\frac{8533\sqrt{3}}{3600}-\frac{14929}{3600}\right)\delta^2. \qquad (49)
$$

The parameter δ is the reduced area, i.e., $\delta=(A-A_c)/A_c$. From (35), we find $F_s'(A)$; the result is

$$
\begin{aligned}
F_s'(A)=\frac{3A}{1024}\Big[\;&35(a^{12}+b^{12}+c^{12})\\
&-12(a^6b^6+a^6c^6+b^6c^6)\\
&+12a^2b^2c^2(a^4b^2+a^4c^2+b^4a^2\\
&+b^4c^2+c^4a^2+c^4b^2)\\
&+14a^4b^4c^4-2a^2b^2c^2(a^6+b^6+c^6)\\
&-19(a^8b^4+a^8c^4+b^8a^4\\
&+b^8c^4+c^8a^4+c^8b^4)\\
&-10(a^{10}b^2+a^{10}c^2+b^{10}a^2+b^{10}c^2\\
&+c^{10}a^2+c^{10}b^2)\;\Big]\\
&+\Big[\;\frac{5}{64}(a^8+b^8+c^8)\\
&-\frac{1}{16}(a^6b^2+a^6c^2+b^6a^2+b^6c^2+c^6a^2+c^6b^2)\\
&+\frac{1}{16}a^2b^2c^2(a^2+b^2+c^2)\\
&-\frac{1}{32}(a^4b^4+a^4c^4+b^4c^4)\;\Big]\int_c^b\frac{\lambda\,\mathrm{d}\lambda}{R(\lambda)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}\Big[\ a^6+b^6+c^6\\
&-(a^2b^4+a^2c^4+b^2a^4+b^2c^4+c^2a^4+c^2b^4)\\
&+2a^2b^2c^2\ \Big]\int_c^b \frac{\lambda^3 \mathrm{d}\lambda}{R(\lambda)}\\
&+\frac{1}{4}\left[\ a^4+b^4+c^4-2(a^2b^2+a^2c^2+b^2c^2)\ \right]\\
&\times\int_c^b \frac{\lambda^5 \mathrm{d}\lambda}{R(\lambda)}+(a^2+b^2+c^2)\int_c^b \frac{\lambda^7 \mathrm{d}\lambda}{R(\lambda)}\\
&-2\int_c^b \frac{\lambda^9 \mathrm{d}\lambda}{R(\lambda)}. \qquad (50)
\end{aligned}
$$

To compute $F'_s(A)$, we express the parameters a, b and c in terms of δ; after a lengthy calculation, (50) reduces to

$$
F'_s(A)=\frac{1}{6A}\left[\ 1+\left(\frac{271}{10800}+\frac{1289\sqrt{3}}{504000}\right)\delta^2+\cdots\ \right]. \qquad (51)
$$

If we compare this with $F'_w(A)$ (given in the begining of this section), we find

$$
\begin{aligned}
&F'_s(A)-F'_w(A)\\
&=\left(\frac{271}{64800}+\frac{1289\sqrt{3}}{3024000}\right)\frac{1}{A_c}\left(\frac{A-A_c}{A_c}\right)^2+\cdots. \qquad (52)
\end{aligned}
$$

Therefore, we have a *third-order* phase transition, which is the same as the ordinary YM_2 ([4]) and the $G(\phi)=\phi^4$ model ([5]). In principle, one could study other ϕ^{2k} models in the same manner; however, while the $k=4$ case can be solved analytically, its expressions become too complicated, and the $k\geq 5$ case cannot be solved analytically.

5 The $G(\phi)=\phi^2+g\phi^4$ model

So far in our study of the phase transition for gYM_2 theories, we have considered only those $G(\phi)$ that contained a single term. In [9], the authors study the phase transition of gYM_2 on a closed surface of arbitrary genus with area A. In particular, they investigate the $G(\phi)=\phi^2+g\phi^3$ model. However, their treatment is mostly qualitative. In this section we consider a simple combination of ϕ^2 and ϕ^4; namely, we study the $G(\phi)=\phi^2+g\phi^4$ model.

In the weak region we can obtain the density ρ from (10); the result is

$$
\rho(z)=\frac{A}{\pi}\sqrt{a^2-z^2}(1+ga^2+2gz^2). \qquad (53)
$$

The above density will have only one maximum at $z=0$, when

$$
3ga^2\leq 1. \qquad (54)
$$

The normalization condition (13) yields

$$
\frac{1}{2}Aa^2+\frac{3}{4}gAa^4=1. \qquad (55)
$$

Using (55), condition (54) reduces to

$$
g\leq A/4. \qquad (56)
$$

Therefore, if this condition is satisfied, we will have a two-cut problem in the $A>A_c$ areas. Here after, we restrict ourselves to this region (condition (56)).

Using (15), we determine the free energy of this model:

$$
F'_w(A)=\frac{1}{8}a^4A+\frac{5}{16}ga^6A+\frac{9}{64}g^2a^8A. \qquad (57)
$$

To study this model in the strong ($A>A_c$) region, we use the following ansatz for ρ ([4]):

$$
\rho_s(z)=\begin{cases}1, & z\in[-b,b]\\ \tilde{\rho}_s(z), & z\in[-a,-b]\bigcup[b,a]\end{cases}. \qquad (58)
$$

Using complex analysis ([4,5,6]), we obtain the function $H_s(z)$, defined using (22); the result is

$$
\begin{aligned}
H_s(z)&=Az+2gAz^3-\sqrt{(z^2-a^2)(z^2-b^2)}\\
&\times\left[2gAz+\int_{-b}^b \frac{\mathrm{d}\lambda}{(z-\lambda)U(\lambda)}\right], \qquad (59)
\end{aligned}
$$

where

$$
U(\lambda)=\sqrt{(a^2-\lambda^2)(b^2-\lambda^2)}. \qquad (60)
$$

Recalling (22), we see that $H_s(z)$ should behave as $1/z$ for large z. Therefore the coefficient of z in (59) must be equal to zero,

$$
A+gAM-\int_{-b}^b \frac{\mathrm{d}\lambda}{U(\lambda)}=0, \qquad (61)
$$

and the coefficient of $1/z$ must be equal to 1:

$$
\frac{1}{2}MA+gA\left(\frac{3}{4}M^2-N\right)-\int_{-b}^b \frac{\lambda^2\mathrm{d}\lambda}{U(\lambda)}=1. \qquad (62)
$$

In the above relations, $M=a^2+b^2$ and $N=a^2b^2$. The two unknown parameters a and b can be determined from these two equations.

Using equations (15), (22) and (59), we obtain the following expression for free energy:

$$
\begin{aligned}
F'_s(A)=&\left(\frac{1}{8}M^2-\frac{1}{2}N\right)\int_{-b}^b \frac{\mathrm{d}\lambda}{U(\lambda)}\\
&+\frac{1}{2}M\int_{-b}^b \frac{\lambda^2\mathrm{d}\lambda}{U(\lambda)}-\int_{-b}^b \frac{\lambda^4\mathrm{d}\lambda}{U(\lambda)}\\
&+g\ \Big[\left(-\frac{1}{4}MN+\frac{1}{16}M^3\right)\int_{-b}^b \frac{\mathrm{d}\lambda}{U(\lambda)}\\
&+\left(\frac{1}{8}M^2-\frac{1}{2}N\right)\int_{-b}^b \frac{\lambda^2\mathrm{d}\lambda}{U(\lambda)}\\
&+\frac{1}{2}M\int_{-b}^b \frac{\lambda^4\mathrm{d}\lambda}{U(\lambda)}-\int_{-b}^b \frac{\lambda^6\mathrm{d}\lambda}{U(\lambda)}\Big]
\end{aligned}
$$

$$-2gA\left(\frac{1}{4}MN-\frac{1}{16}M^3\right)$$
$$-2g^2A\left(-\frac{1}{8}N^2+\frac{3}{16}M^2N-\frac{5}{128}M^4\right). \quad (63)$$

In order to investigate the phase transition, we use the following change of variables:

$$a=a_c(1-h), \quad (64)$$

and

$$\left(\frac{b}{a}\right)^2=k. \quad (65)$$

The parameter a_c is the value of a at the critical point, and h and k are equal to zero at this point. At the critical point, the normalization condition (55) becomes

$$\frac{1}{2}A_c{a_c}^2+\frac{3}{4}gA_c{a_c}^4=1. \quad (66)$$

At $A=A_c$, $\rho(0)$ is equal to one, yielding

$$a_cA_c+gA_c{a_c}^3=\pi. \quad (67)$$

In order to study the phase transition of this model, we limit ourselves to the small values of g (where $g\ll 1$). Using (66) and (67), A_c and a_c then become

$$A_c=\frac{\pi^2}{2}-g+\frac{4g^2}{\pi^2}+\cdots, \quad (68)$$

and

$$a_c=\frac{2}{\pi}-\frac{4g}{\pi^3}+\frac{40g^2}{\pi^5}+\cdots. \quad (69)$$

Now we follow the same steps used in Sect. 4. First, we expand equations (61) and (62) in terms of h and k, and solve these equations for h and k. The results, up to order δ^2, are

$$h=(\frac{1}{2}-\frac{3}{2}\frac{g}{A_c}+\frac{27}{2}\frac{g^2}{A_c^2})\delta$$
$$+(-\frac{5}{8}+\frac{7}{4}\frac{g}{A_c}-\frac{287}{8}\frac{g^2}{A_c^2})\delta^2+\cdots, \quad (70)$$

and

$$k=(2+14\frac{g}{A_c}+42\frac{g^2}{A_c^2})\delta$$
$$+(-\frac{7}{4}-\frac{93}{2}\frac{g}{A_c}+\frac{283}{4}\frac{g^2}{A_c^2})\delta^2+\cdots. \quad (71)$$

Second, we find $F_s'(A)$ using equation (63). The result is

$$F_s'(A)=\frac{1}{A}\Big\{\frac{1}{2}-\frac{1}{2}\frac{g}{A}+\frac{9}{4}\frac{g^2}{A^2}$$
$$+\left(1+3\frac{g}{A}+85\frac{g^2}{A^2}\right)\delta^2+\cdots\Big\}. \quad (72)$$

This relation is true only up to order g^2 and δ^2. To compare (72) with $F'(A)$ in the weak region, we first note that a^2, up to order g^2, is (from (55))

$$a^2=\frac{2}{A}\left(1-3\frac{g}{A}+18\frac{g^2}{A^2}\right), \quad (73)$$

and from this, $F_w'(A)$ becomes (from (57)):

$$F_w'(A)=\frac{1}{A}\left(\frac{1}{2}-\frac{1}{2}\frac{g}{A}+\frac{9}{4}\frac{g^2}{A^2}\right). \quad (74)$$

Therefore

$$F_s'(A)-F_w'(A)$$
$$=\frac{1}{A_c}\left(1+3\frac{g}{A_c}+85\frac{g^2}{A_c^2}\right)\left(\frac{A-A_c}{A_c}\right)^2+\cdots. \quad (75)$$

This shows that the $G(\phi)=\phi^2+g\phi^4$ model in $g<A/4$ region also has a *third-order* phase transition. Also note that at $g=0$, (75) reduces to

$$F_s'(A)-F_w'(A)=\frac{2}{\pi^2}\left(\frac{A-A_c}{A_c}\right)^2+\cdots, \quad (76)$$

which is the same relation obtained in [4] for the ordinary 2-dimensional Yang-Mills theory.

Finally, it should be noted that for $g>A/4$, the density function ρ in (53) has two symmetric maxima in the weak ($A<A_c$) region. This gives us a three-cut problem and the above perturbative calculation is not possible.

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